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Invariant curves for a delay differential equation with a piecewise constant argument

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In order to understand the dynamics of a second order delay differential equation with a piecewise constant argument, we investigate invariant curves of the derived planar mapping from the equation. All invariant curves are given in this paper.

Keywords: difference equation; invariant curve; piecewise construction; characteristic root; dual equation

1 Introduction

The study of differential equations with piecewise constant argument (EPCA) initiated in [1, 2]. These equations represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations, hence, they are of importance in control theory and in certain biomedical models [3]. In this paper the second order delay differential equation with a piecewise constant argument

$$x''(t) + g(x([t])) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}, \quad (1)$$

where $x''(t)$ denotes the second order derivative of $x(t)$, $[t]$ denotes the greatest integer less than or equal to t , and $g: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous or at least piecewise continuous function, is considered. In 1987, Aftabizadeh *et al.* discussed the oscillatory and periodic properties of the solutions of (1) in [4]. In 1989, Gyori and Ladas investigated linearized oscillations of the solutions of (1) in [5]. Later, Wiener and Cooke considered oscillations of the solutions of systems of two differential equations with piecewise constant arguments in [6].

The invariant curve [7–11] is another interesting problem in the study of dynamics because it can be used to reduce a system to a 1-dimensional one. The problem of invariant curves is actually a part of the research on invariant manifolds. In 1997, Ng and Zhang studied the nonlinear C^1 invariant curve of planar mapping $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$,

$$G(x, y) = \left(y, 2y - x - \frac{1}{2}(g(y) + g(x)) \right), \quad (2)$$

derived from (1) in [12] when g is nonlinear and gave the conditions that G has linear invariant curves when g is linear. In 2003, Yang *et al.* investigated nonlinear C^0 invariant curves of (2) when g is nonlinear in [13]. So far, nonlinear invariant curves of (2) when g is linear have not been studied. So it is very interesting to look for nonlinear invariant curves

of (2) when g is linear. In this paper all the invariant curves of the planar mapping G are given including the linear and nonlinear ones when g is linear.

2 Main results

We discuss invariant curves of the form $y = f(x)$ for the planar mapping (2). Its invariant curves of the form $y = f(x)$ satisfy $f(y) = 2y - x - \frac{1}{2}(g(y) + g(x))$, which leads to the iterative functional equation

$$f(f(x)) = 2f(x) - x - \frac{1}{2}(g(f(x)) + g(x)), \quad \forall x \in \mathbb{R}. \quad (3)$$

Considering linear g and $g(x) = ax + b$, we compute that

$$f(f(x)) - \left(2 - \frac{a}{2}\right)f(x) + \left(1 + \frac{a}{2}\right)x = -b, \quad \forall x \in \mathbb{R}. \quad (4)$$

Thus, the invariant curves of planar mapping G with $g(x) = ax + b$ can be obtained by solving functional (4). We mainly discuss the generic cases $a \notin \{-2, 4\}$, but leave the special cases $a = -2$ and $a = 4$ to the last part of this section. For generic $a \notin \{-2, 4\}$, (4) with $b = 0$ is of the form discussed in [14, 15]. In order to apply the results of [14], we let

$$r_1 := \frac{(4-a) - (a^2-16a)^{\frac{1}{2}}}{4}, \quad r_2 := \frac{(4-a) + (a^2-16a)^{\frac{1}{2}}}{4}, \quad (5)$$

which are the roots of the characteristic polynomial $P(r) := r^2 - (2 - \frac{a}{2})r + 1 + \frac{a}{2}$.

From (5) we see that the characteristic roots r_1, r_2 of (4) have the following possibilities:

- (C1) $0 < r_1 < 1 < r_2$, if and only if $-2 < a < 0$.
- (C2) $r_1 = r_2 = 1$, if and only if $a = 0$.
- (C3) $r_1 < 0 < r_2 \neq 1$ and $r_1 \neq -r_2$, if and only if $a < -2$.
- (C4) $r_1 = r_2 < 0$, if and only if $a = 16$.
- (C5) $r_1 < r_2 < -1$, if and only if $a > 16$.

Note that the case $r_2 > r_1 > 1$ is not listed because the case $r_2 > r_1 > 1$ implies $\frac{(4-a) - (a^2-16a)^{\frac{1}{2}}}{4} > 1$, i.e., $-a > (a^2-16a)^{\frac{1}{2}}$, which does not hold, and that the case $0 < r_1 < r_2 < 1$ is not listed because $0 < r_1 < r_2 < 1$ implies $0 < \frac{(4-a) + (a^2-16a)^{\frac{1}{2}}}{4} < 1$, i.e., $0 < a < 4$, which contradicts the requirement that $\Delta = a^2 - 16a \geq 0$, and that the case $0 < a < 16$ is not listed because in this case (4) with $b = 0$ has no continuous solutions, neither r_1 nor r_2 is real, by [14]. Since we consider $a \notin \{-2, 4\}$, none of the case $r_1 = 0$, the case $r_2 = 0$, and the case $r_1 = -r_2 \neq 0$ is listed. Corresponding to the above list, we have the following results.

Theorem 2.1 (i) If $-2 < a < 0$, then a continuous solutions ϕ of (4) with $b = 0$ is either of the piecewise linear form that $f(x) := r_1 x$ for $x > 0$, or $:= 0$ for $x = 0$, or $:= r_j x$ for $x < 0$, where $i, j = 1, 2$, or given by

$$f(x) := \begin{cases} f_n(x), & x \in [x_n, x_{n+1}), n = 0, 1, 2, \dots, \\ f_{-n}^{-1}(x), & x \in [x_{-n}, x_{-n+1}), n = 1, 2, \dots, \end{cases}$$

where $x_n = \frac{r_2^n}{r_2 - r_1}(x_1 - r_1 x_0) + \frac{r_1^n}{r_2 - r_1}(-x_1 + r_2 x_0)$, $n \in \mathbb{Z}$, with an arbitrarily chosen $x_0 \in (-\infty, +\infty)$ and $x_1 \in [r_1 x_0, r_2 x_0]$, and $f_n(x) = (r_1 + r_2)x - r_1 r_2 f_{n-1}^{-1}(x)$ for all $x \in [x_n, x_{n+1})$, $n = 1, 2, \dots$, $f_{-n-1}(x) = (\frac{1}{r_1} + \frac{1}{r_2})x - \frac{1}{r_1 r_2} f_{-n}^{-1}(x)$ for all $x \in [x_{-n}, x_{-n+1})$, $n = 1, 2, \dots$, and $f_{-1}(x) =$

$(\frac{1}{r_1} + \frac{1}{r_2})x - \frac{1}{r_1 r_2} f_0(x)$, $x \in [x_0, x_1]$, with the arbitrarily chosen functions f_0 such that $f_0(x_0) = x_1$, $f_0(x_1) = x_2$, and $r_1 \leq \frac{f_0(x) - f_0(y)}{x - y} \leq r_2$ for all $x, y \in [x_0, x_1]$. (ii) If $a = 0$, then (4) with $b = 0$ has a unique continuous solution f and $f(x) = x + \beta$, where $\beta \in \mathbb{R}$ is an arbitrary constant.

Proof The proof is a simple application of well-known results in [14]. The result (i) is given by Theorem 2 of [14], where the characteristic roots r_1, r_2 satisfy $r_2 > 1 > r_1 > 0$ as shown in (C1). We can deduce the result (ii) from Theorem 8 of [14], where $r_1 = r_2 = 1$ as shown in (C2). The proof is completed. \square

Theorem 2.2 (i) If $a < -2$, then (4) with $b = 0$ only has two continuous solutions f and $f(x) = r_1 x$ or $r_2 x$. (ii) If $a = 16$, (4) with $b = 0$ just has a continuous solution $f(x) = -3x$. (iii) If $a > 16$, all continuous solutions f of (4) with $b = 0$ are given by

$$f(x) := \begin{cases} f_{2n}(x), & x \in [x_{-2n}, x_{-2n+2}), n = 0, 1, 2, \dots, \\ f_{2n+1}(x), & x \in [x_{-2n+3}, x_{-2n+1}), n = 0, 1, 2, \dots, \\ 0, & x = 0, \\ f_{-2n}^{-1}(x), & x \in [x_{2n}, x_{2n+2}), n = 1, 2, \dots, \\ f_{-2n+1}^{-1}(x), & x \in [x_{2n+3}, x_{2n+1}), n = 1, 2, \dots, \end{cases}$$

where the sequence $\{x_n\}$ is defined by $x_n = \frac{r_2^n}{r_2 - r_1}(x_1 - r_1 x_0) + \frac{r_1^n}{r_2 - r_1}(-x_1 + r_2 x_0)$, $n \in \mathbb{Z}$, with an arbitrarily chosen $x_0 \in (0, +\infty)$ and $x_1 \in [r_1 x_0, r_2 x_0]$, and $f_{2n-1}(x) = (r_1 + r_2)x - r_1 r_2 f_{2n-2}^{-1}(x)$, $x \in [x_{-2n+5}, x_{-2n+3})$, $n = 1, 2, \dots$, $f_{2n}(x) = (r_1 + r_2)x - r_1 r_2 f_{2n-1}^{-1}(x)$, $x \in [x_{-2n}, x_{-2n+2})$, $n = 1, 2, \dots$, $f_{-2n}(x) = (\frac{1}{r_1} + \frac{1}{r_2})x - \frac{1}{r_1 r_2} f_{-2n+1}^{-1}(x)$, $x \in [x_{2n+3}, x_{2n+1})$, $n = 1, 2, \dots$, $f_{-2n-1}(x) = (\frac{1}{r_1} + \frac{1}{r_2})x - \frac{1}{r_1 r_2} f_{-2n}^{-1}(x)$, $x \in [x_{2n}, x_{2n+2})$, $n = 1, 2, \dots$, and $f_{-1}(x) = (\frac{1}{r_1} + \frac{1}{r_2})x - \frac{1}{r_1 r_2} f_0(x)$, $x \in [x_0, x_2]$, with an arbitrarily chosen continuous function f_0 on $[x_0, x_2]$ such that $f_0(x_0) = x_1$, $f_0(x_2) = x_3$, and $r_1 \leq \frac{f_0(x) - f_0(y)}{x - y} \leq r_2$, $\forall x, y \in [x_0, x_2]$.

Proof Firstly, we consider (i). By Theorem 5 in [14], (4) with $b = 0$ only has two continuous solutions f and $f(x) = r_1 x$ or $r_2 x$, where the characteristic roots r_1, r_2 satisfy $r_1 < 0 < r_2 \neq 1$ and $r_1 \neq -r_2$ as shown in (C3). Next, we consider (ii). By Theorem 6 in [14], (4) with $b = 0$ just has a continuous solution $f(x) = -3x$, where the characteristic roots r_1, r_2 satisfy $r_1 = r_2 = -3$ as shown in (C4). Finally, we consider (iii). In order to piecewise construct all solutions of (4) with $b = 0$ we need a partition for the interval $(-\infty, \infty)$. For this purpose we consider a homogeneous linear difference equation

$$x_{n+2} - \left(2 - \frac{a}{2}\right)x_{n+1} + \left(1 + \frac{a}{2}\right)x_n = 0, \quad (6)$$

which has the same coefficients as (4) with $b = 0$ correspondingly. Its characteristic equation is

$$r^2 - \left(2 - \frac{a}{2}\right)r + 1 + \frac{a}{2} = 0, \quad (7)$$

which has two characteristic roots r_1 and r_2 satisfying $r_1 < r_2 < -1$ as shown in (C5). Thus, (4) and (6) can be, respectively, rewritten as

$$f(f(x)) - (r_1 + r_2)f(x) + r_1 r_2 x = 0, \quad (8)$$

$$x_{n+2} - (r_1 + r_2)x_{n+1} + r_1 r_2 x_n = 0, \quad n = 0, 1, 2, \dots \quad (9)$$

If f is a solution of (4) with $b = 0$, we easily see that f is invertible. In fact, if $f(x_1) = f(x_2)$, then $f(f(x_1)) = f(f(x_2))$. Thus, $x_1 = x_2$ by (4) because $a \neq -2$, which implies that f is one to one. Next we only need to show that $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$ and $f(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ because $f(x) \rightarrow \pm\infty$ as $x \rightarrow \pm\infty$, then the left-hand side of (4) with $b = 0$ tends to $\pm\infty$ by $a > 16$, but the right-hand side is equal to 0. Otherwise, $f(x)$ has a finite limit as $x \rightarrow \infty$, then $f(f(x)) - (2 - \frac{a}{2})f(x)$ converges to a finite limit by the continuity of f on the whole of \mathbb{R} , but $(1 + \frac{a}{2})x$ does not, which contradicts the requirement that $f(f(x)) - (2 - \frac{a}{2})f(x) = -(1 + \frac{a}{2})x$. Thus, we rewrite (4) in the following equivalent form:

$$f^{-1}(f^{-1}(x)) - \left(2 - \frac{a}{2}\right)f^{-1}(x) + \frac{a}{2}x = 0, \quad (10)$$

which is called the dual equation to (4) with $b = 0$. Solving the homogeneous linear difference (9) with arbitrarily chosen real initial values x_0 and x_1 , we obtain

$$x_n = \frac{r_2^n}{r_2 - r_1}(x_1 - r_1x_0) + \frac{r_1^n}{r_2 - r_1}(-x_1 + r_2x_0), \quad n \in \mathbb{Z}. \quad (11)$$

Let $x_0 = x$ and $x_{n+1} = f(x_n)$ in (11), we have

$$f^n(x) = \frac{r_2^n}{r_2 - r_1}(f(x) - r_1x) + \frac{r_1^n}{r_2 - r_1}(-f(x) + r_2x), \quad n \in \mathbb{Z}.$$

Furthermore, we can obtain

$$\Delta f^n(x, y) = \frac{r_2^n}{r_2 - r_1}(\Delta f(x, y) - r_1) + \frac{r_1^n}{r_2 - r_1}(-\Delta f(x, y) + r_2), \quad (12)$$

$$f^{n+1}(x) - f^n(x) = r_2^n \frac{r_2 - 1}{r_2 - r_1}(f(x) - r_1x) + r_1^n \frac{r_1 - 1}{r_2 - r_1}(-f(x) + r_2x), \quad (13)$$

where $\Delta f^n(x, y) = \frac{f^n(x) - f^n(y)}{x - y}$ for any $x \neq y$ and $n \in \mathbb{Z}$. From (12) we can see that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{\Delta f^n(x, y)}{r_2^n} &= \frac{(\Delta f(x, y) - r_1)}{r_2 - r_1}, \\ \lim_{n \rightarrow -\infty} \frac{\Delta f^n(x, y)}{r_1^n} &= \frac{(-\Delta f(x, y) + r_2)}{r_2 - r_1}. \end{aligned}$$

Since f is strictly monotonic, $\Delta f^n(x, y) > 0$ for even n , which implies $\Delta f(x, y) - r_1 \geq 0$ and $-\Delta f(x, y) + r_2 \geq 0$, that is,

$$r_1 \leq \Delta f(x, y) \leq r_2. \quad (14)$$

Moreover, we can see that $f(0) = 0$ from (13). In what follows, we arbitrarily choose $x_0 \in (0, +\infty)$ and $x_1 \in [r_1x_0, r_2x_0]$ and define a sequence $\{x_n\}$, $n \in \mathbb{Z}$, by (11). The sequences $\{x_{2n}\}$, $\{x_{2n+1}\}$, $\{x_{-2n}\}$ and $\{x_{-2n+1}\}$, where $n = 0, 1, 2, \dots$, are strictly monotone such that $x_{2n} \rightarrow +\infty$, $x_{2n+1} \rightarrow -\infty$, $x_{-2n} \rightarrow 0$, and $x_{-2n+1} \rightarrow 0$ as $n \rightarrow \infty$. Thus, the sequence $\{x_n\}$, $n \in \mathbb{Z}$, is a partition of the interval $(-\infty, \infty)$. Next we arbitrarily choose a continuous function defined in the interval $[x_0, x_2]$, satisfying $f_0(x_0) = x_1$, $f_0(x_2) = x_3$, and condition (14). We can

recursively define the homeomorphisms $f_{2n-1} : [x_{-2n+5}, x_{-2n+3}] \rightarrow [x_{-2n}, x_{-2n+2}]$, $n = 1, 2, \dots$, and $f_{2n} : [x_{-2n}, x_{-2n+2}] \rightarrow [x_{-2n+3}, x_{-2n+1}]$, $n = 1, 2, \dots$, such that

$$f_{2n-1}(x_{-2n+5}) = x_{-2n+2}, \quad f_{2n-1}(x_{-2n+3}) = x_{-2n}, \quad (15)$$

$$f_{2n}(x_{-2n}) = x_{-2n+1}, \quad f_{2n}(x_{-2n+2}) = x_{-2n+3}, \quad (16)$$

$$r_1 \leq \Delta f_{2n-1}(x, y) \leq r_2, \quad \forall x, y \in [x_{-2n+5}, x_{-2n+3}], \quad (17)$$

$$r_1 \leq \Delta f_{2n}(x, y) \leq r_2, \quad \forall x, y \in [x_{-2n}, x_{-2n+2}]. \quad (18)$$

In fact, for f_{2n} defined satisfying (16) and (18), we let

$$f_{2n+1}(x) = (r_1 + r_2)x - r_1 r_2 f_{2n}^{-1}(x), \quad \forall x \in [x_{-2n+3}, x_{-2n+1}].$$

Obviously, $f_{2n+1}(x_{-2n+3}) = x_{-2n}$ and $f_{2n+1}(x_{-2n+1}) = x_{-2n-2}$. Making use of (18), we have $\frac{1}{r_2} \leq \frac{f_{2n}^{-1}(x) - f_{2n}^{-1}(y)}{x - y} \leq \frac{1}{r_1}$ for $x, y \in [x_{-2n}, x_{-2n+2}]$. It is easy to deduce that

$$r_1 \leq \Delta f_{2n+1}(x, y) \leq r_2, \quad \forall x, y \in [x_{-2n+3}, x_{-2n+1}].$$

Furthermore, we again let

$$f_{2n+2}(x) = (r_1 + r_2)x - r_1 r_2 f_{2n+1}^{-1}(x), \quad \forall x \in [x_{-2n-2}, x_{-2n}].$$

By the same argument we can see that

$$f_{2n+2}(x_{-2n-2}) = x_{-2n-1}, \quad f_{2n+2}(x_{-2n}) = x_{-2n+1}, \quad (19)$$

$$r_1 \leq \Delta f_{2n+2}(x, y) \leq r_2, \quad \forall x, y \in [x_{-2n-2}, x_{-2n}]. \quad (20)$$

By induction both f_{2n-1} and f_{2n} are well defined. Similarly, we can also recursively define the homeomorphisms $f_{-2n+1} : [x_{2n-2}, x_{2n}] \rightarrow [x_{2n+3}, x_{2n+1}]$, $n = 1, 2, \dots$, and $f_{-2n} : [x_{2n+3}, x_{2n+1}] \rightarrow [x_{2n}, x_{2n+2}]$, $n = 1, 2, \dots$. By the properties of the dual (10) we can obtain

$$f_{-2n+1}(x_{2n-2}) = x_{2n+1}, \quad f_{-2n+1}(x_{2n}) = x_{2n+3},$$

$$f_{-2n}(x_{2n+3}) = x_{2n+2}, \quad f_{-2n}(x_{2n+1}) = x_{2n},$$

$$\frac{1}{r_2} \leq \Delta f_{-2n+1}(x, y) \leq \frac{1}{r_1}, \quad \forall x, y \in [x_{2n-2}, x_{2n}],$$

$$\frac{1}{r_2} \leq \Delta f_{-2n}(x, y) \leq \frac{1}{r_1}, \quad \forall x, y \in [x_{2n+3}, x_{2n+1}].$$

Therefore,

$$f_{-2n+1}^{-1}(x_{2n+1}) = x_{2n-2}, \quad f_{-2n+1}^{-1}(x_{2n+3}) = x_{2n},$$

$$f_{-2n}^{-1}(x_{2n+2}) = x_{2n+3}, \quad f_{-2n}^{-1}(x_{2n}) = x_{2n+1},$$

$$r_1 \leq \Delta f_{-2n+1}^{-1}(x, y) \leq r_2, \quad \forall x, y \in [x_{2n+3}, x_{2n+1}],$$

$$r_1 \leq \Delta f_{-2n}^{-1}(x, y) \leq r_2, \quad \forall x, y \in [x_{2n}, x_{2n+2}].$$

Thus, we can define

$$f(x) := \begin{cases} f_{2n}(x), & x \in [x_{-2n}, x_{-2n+2}), n = 0, 1, 2, \dots, \\ f_{2n+1}(x), & x \in [x_{-2n+3}, x_{-2n+1}), n = 0, 1, 2, \dots, \\ 0, & x = 0, \\ f_{-2n}^{-1}(x), & x \in [x_{2n}, x_{2n+2}), n = 1, 2, \dots, \\ f_{-2n+1}^{-1}(x), & x \in [x_{2n+3}, x_{2n+1}), n = 1, 2, \dots \end{cases}$$

f is continuous on \mathbb{R} because $f_{2n}(x_{-2n}) = x_{-2n+1} = f_{2n+2}(x_{-2n})$, $f_{2n+1}(x_{-2n+1}) = x_{-2n-2} = f_{2n+3}(x_{-2n+1})$, where $n = 0, 1, 2, \dots$, $f_1^{-1}(x_0) = f_{-1}(x_0)$, $f_{-2n}^{-1}(x_{2n+2}) = x_{2n+3} = f_{-2n-2}^{-1}(x_{2n+2})$, and $f_{-2n+1}^{-1}(x_{2n+3}) = x_{2n+2} = f_{-2n-1}^{-1}(x_{2n+3})$, where $n = 1, 2, 3, \dots$ we can easily check that f defined in Theorem 2.2 satisfies (4) with $b = 0$ in \mathbb{R} . In fact, if $x \in [x_{-2n}, x_{-2n+2})$, $n = 0, 1, 2, \dots$, $f^2(x) = f_{2n+1}(f_{2n}(x)) = (r_1 + r_2)f_{2n}(x) - r_1r_2x = (r_1 + r_2)f(x) - r_1r_2x$, i.e., $f^2(x) - (r_1 + r_2)f(x) - r_1r_2x = 0$. Similarly, we can also check that f satisfies (4) with $b = 0$ for $x \in [x_{-2n+3}, x_{-2n+1})$, $x \in [x_{2n}, x_{2n+2})$ and $x \in [x_{2n+3}, x_{2n+1})$, where $n = 1, 2, 3, \dots$. The proof is completed. \square

Remark In the case that $b \neq 0$, as indicated in [16] for (2) therein, (4) can be reduced equivalently to the equation

$$\tilde{f}(\tilde{f}(x)) - \left(2 - \frac{a}{2}\right)\tilde{f}(x) + \left(1 + \frac{a}{2}\right)x = 0, \quad (21)$$

the same type of equation as the one considered in Theorems 2.1 and 2.2 with vanishing b , by the replacement $\tilde{f}(x) = f(x + \xi) - \xi$, where $\xi = \frac{-b}{(1-r_1)(1-r_2)}$, if its characteristic roots r_1, r_2 are both real but neither of them is equal to 1. In this case solutions can be found from Theorems 2.1 and 2.2. So (4) with $b \neq 0$ can be reduced to (21) except for the case $a = 0$. For the case of $a = 0$ and $b \neq 0$, (4) has no real continuous solutions. In fact, by induction and (4) we can obtain $f^n(x) = nf(x) - (n-1)x - \frac{n(n+1)}{2}b$, $n \in \mathbb{Z}$. Furthermore, we have $f^{n+1}(x) - f^n(x) = f(x) - x - (n+1)b$, $n \in \mathbb{Z}$. For an arbitrary $x \in \mathbb{R}$, $f^{n+1}(x) - f^n(x)$ has the same sign when n takes the values N and $-N$, where N is a large positive integer, because f is strictly monotonic. But $f(x) - x - (n+1)b$ has not, which contradicts the requirement $f^{n+1}(x) - f^n(x) = f(x) - x - (n+1)b$, $n \in \mathbb{Z}$.

In what follows, we consider the case that either $a = -2$ or $a = 4$, which is not generic.

For $a = -2$, (4) is of the form $f^2(x) - 3f(x) = -b$, from which we get with the replacement $y = f(x)$: $f(x) = 3x - b$.

For $a = 4$, (4) is of the form

$$f^2(x) = -3x - b, \quad (22)$$

which is the problem of iterative roots of the linear function $F(x) := -3x - b$. By the theory of iterative roots, as shown in [8], we know (22) has no real continuous solutions.

Competing interests

The author declares that he has no competing interests.

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